Name: ______________________ Teacher: ____________________

GOSFORD HIGH SCHOOL

MATHEMATICS EXTENSION 2

2014

HSC Assessment Task #2

Time Allowed: 90 minutes (plus 5 minutes reading)

- Answer **SECTION I** (multiple choice) on the answer sheet provided – no working needs to be shown.
- **SECTION II:** Should be attempted on your own paper, starting each question on a new sheet of paper – all necessary working **MUST** be shown.
- Write using black or blue pen and board approved calculators may be used.

SECTION	QUESTIONS	MARKS	RESULT
I	Question 1–5: Multiple Choice	5	
II	Question 6: Complex Numbers	15	
	Question 7: Graphs	15	
	Question 8: Polynomials	15	
	TOTAL	50	

SECTION I: (5 marks)

- **Attempt Questions 1 – 5**
- **Allow about 10 minutes for this section**
- **Use the multiple choice answer sheet for Questions 1 – 5**

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1. Which of the following **CANNOT** be the argument of a complex number z such that $z^9 = -1 + i$

(A) $\frac{11\pi}{36}$ (B) $\frac{\pi}{12}$

(C) $\frac{29\pi}{36}$ (D) $\frac{19\pi}{36}$

2. If $P(z) = z^3 + pz + q$,
where p, q are real and one root of
$P(z) = 0$ is 1 – i, then the real root is

(A) – 2

(B) $\frac{1}{2}$

(C) 2

(D) Cannot be determined

3. The graph of $f(x) = \frac{1}{x^2+mx-n}$,

where *m and n* are real constants, has no vertical asymptotes if:

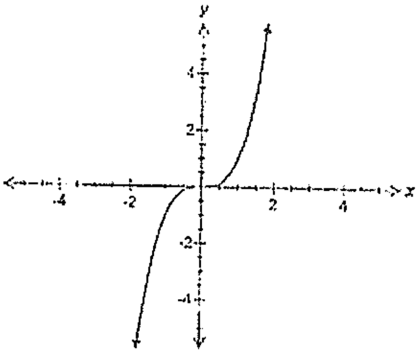
(A) $m^2 < 4n$

(B) $m^2 > 4n$

(C) $m^2 = -4n$

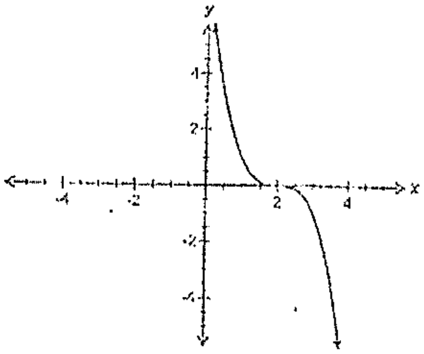
(D) $m^2 < -4n$

4. The graph of $y = f(x)$ is shown below:

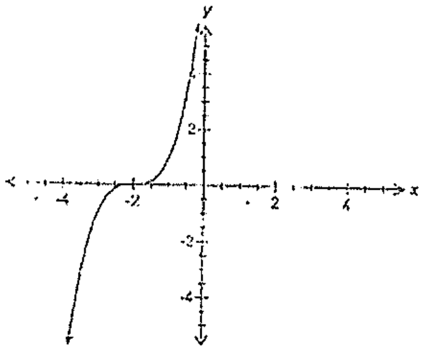


Which graph best represents $y = f(2-x)$?

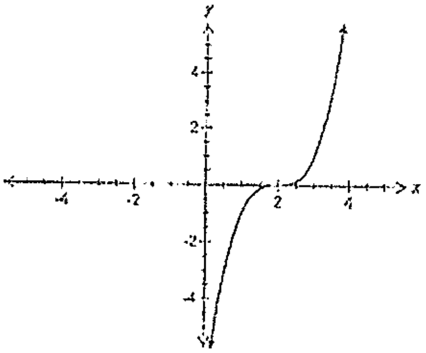
(A)



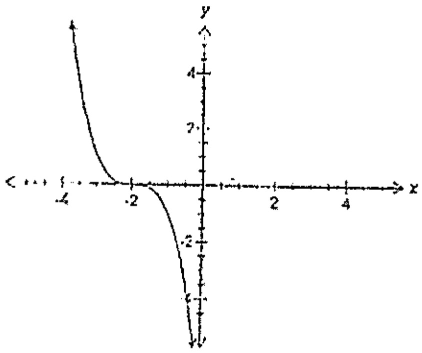
(B)



(C)



(D)



5. What restrictions must be placed on p if there are three, non-zero real roots of the equation:

$x^3 + px - 1 = 0$?

(A) p > 0, p is real

(B) p < 0, p is real

(C) p$\geq$ 0, p is real

(D) p$\leq$ 0, p is real

END OF SECTION I

SECTION II: (45 marks)

- **Attempt Questions 6 – 8**
- **Allow about 1 hour 20 minutes for this section**
- Answer questions on your own A4 paper, starting each question on a new sheet of paper.
- In questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations

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Question 6: (Complex Number) (15 Marks) *Use a new sheet of paper.*

a. If $z = 3 - i$ and $w = 2i - 1$ then find 1
$w^2 - 2\overline{z}$

b. (i) Express $z = \sqrt{3} + i$ in modulus/argument form 3

(ii) Hence, show that

$z^7 + 64z = 0$

c. Find the complex square roots of $7 - 6\sqrt{2}i$, 3

giving your answers in form $x + iy$, where x, y are real.

d. Sketch the region of the Argand Diagram whose points satisfy the inequalities 2

$|z - \overline{z}| \leq 4$ <u>and</u> $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$

e. ABCD is a quadrilateral whose diagonals AC and BD are equal and bisect each other at the origin. A is the complex number z and ∠AOB = 30°

Find possible co-ordinates of B, C, D in terms of z 2

f. (i) Indicate vectors representing z and $z - 2$ 1
in an Argand Diagram so that
$\arg(z - 2) = 2 \arg z$

(ii) If the point P represents z, O is the origin and Q has co-ordinates (2,0) in 3
this Argand Diagram, what type of triangle is Δ OPQ for non-real z?
Also, deduce that if z is non-real, then P lies on a circle and state its centre and radius.

END OF QUESTION 6

Question 7: Graphs (15 Marks) Start *a new sheet of paper.*

(For this question neat sketches should be drawn on approximately 1/3 page number planes.)

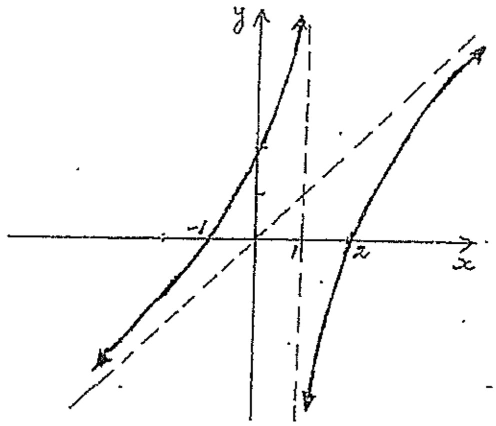
a. Consider the function $f(x) = \frac{e^x-1}{e^x+1}$

- i. Show that $f(x)$ is an odd function — 1
- ii. Show that the function is always increasing. — 2
- iii. Find $f'(o)$ — 1
- iv. Discuss the behaviour of $f(x)$ as $x \to \pm\infty$ — 1
- v. Hence, sketch the graph of $y = f(x)$ — 1

b. The function $y = f(x)$ is denoted by $f(x) = x^3 - 4x$

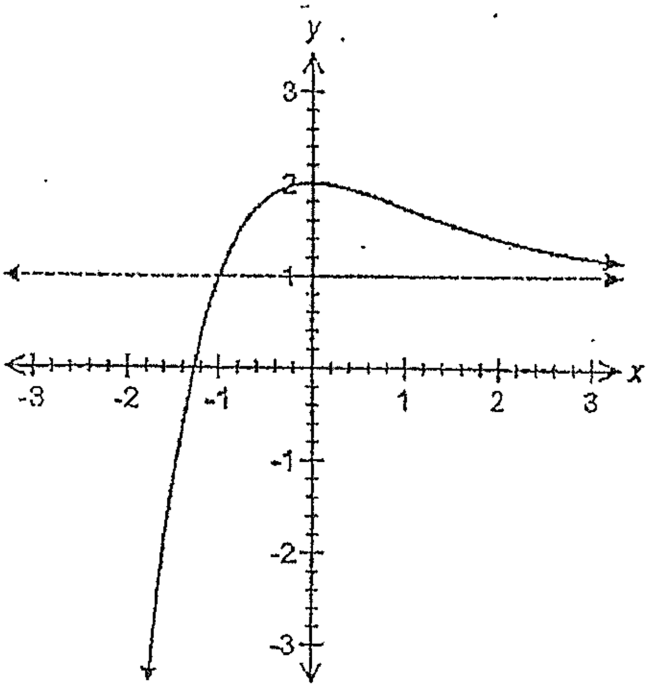
- i. Sketch the graph of $y = f(x)$ indicating all intercepts. — 1
- ii. On a separate number plane, sketch the graph of $y = \frac{1}{x^3-4x}$ — 1

c. The sketch of $y = f(x)$ is shown below, where $f(x) = \frac{x^2-x-2}{x-1}$



- i. Prove that $y = x$ is an asymptote — 2
- ii. Sketch each of the following on the template sheet:
  - (∝) $y = |f(x)|$ — 1
  - (β) $y = \sqrt{f(x)}$ — 1
  - (γ) $y^2 = f(x)$ — 1

d. The diagram shows the graph $y = f(x)$



Draw separate sketches of the following on the template sheet provided.

i. $y = f(|x|)$ 1

ii. $y = \ln[f(x)]$ 1

END OF QUESTION 7

Question 8: (Polynomials) (15 Marks) Start *a new sheet of paper.*

a. Express $\frac{x+7}{x^2(x+2)}$ in the form: 2

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$$

b. Calculate the product of the roots of 2

$$(3+2i)z^2-(1-2i)z+\ (6-i)=0$$

in the form $a+ib$

c. *If* $P(x)=2x^3-3x^2+4x+3$ and 2
given that $P\left(-\frac{1}{2}\right)=0$, fully factorise
$P(x)$ over the complex field.

d. If the cubic equation 2

$$2y^3-9y^2+12y+k=0$$

has two equal roots, find the possible value(s) of k

e. Given that the roots of $x^3+2x^2-3x-1=0$ are $\propto, \beta$ and γ, 3

find an equation whose roots are $\frac{1}{\propto^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

f. Let $w=\cos\frac{2\pi}{7}+isin\frac{2\pi}{7}$

i. Show that w^k is a solution of $z^7-1=0$, where k is an integer. 1

ii. Prove that $w+\ w^2+w^3+\ w^4+\ w^5+\ w^6=\ -1$ 1

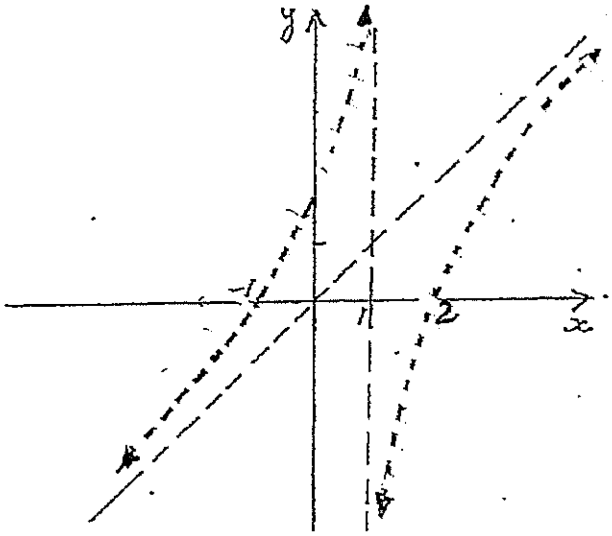
iii. Hence, show that 2

$$cos\frac{2\pi}{7}+\ cos\frac{4\pi}{7}+\ cos\frac{6\pi}{7}=\ -\frac{1}{2}$$

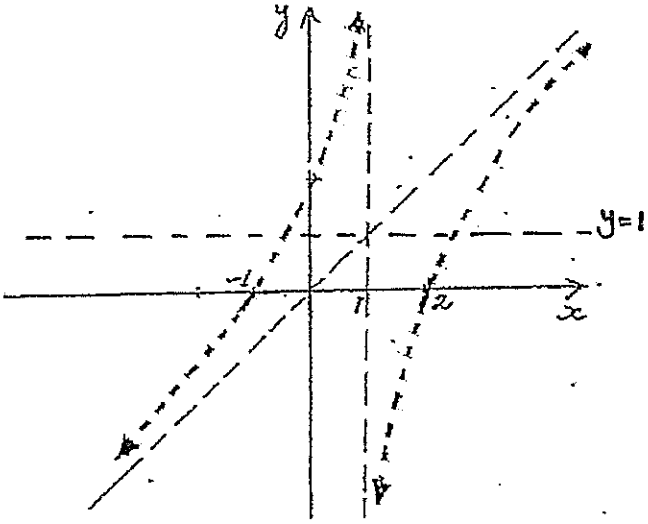
END OF ASSESSMENT TASK

Template for Question 7 (c)(ii)

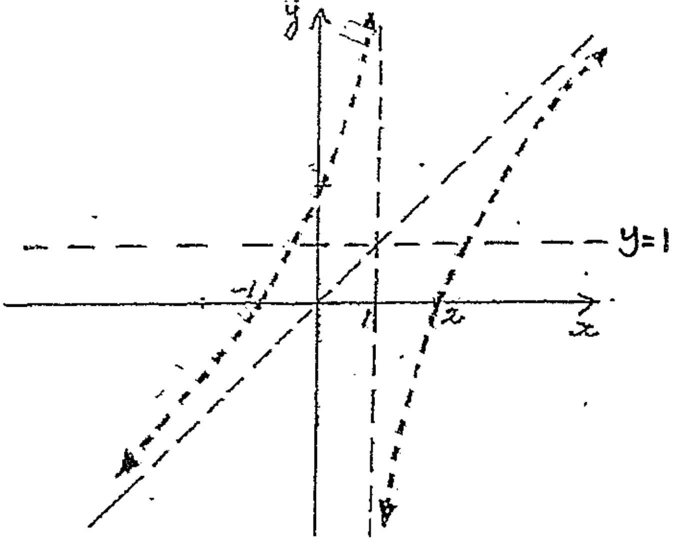
(∝)



(β)

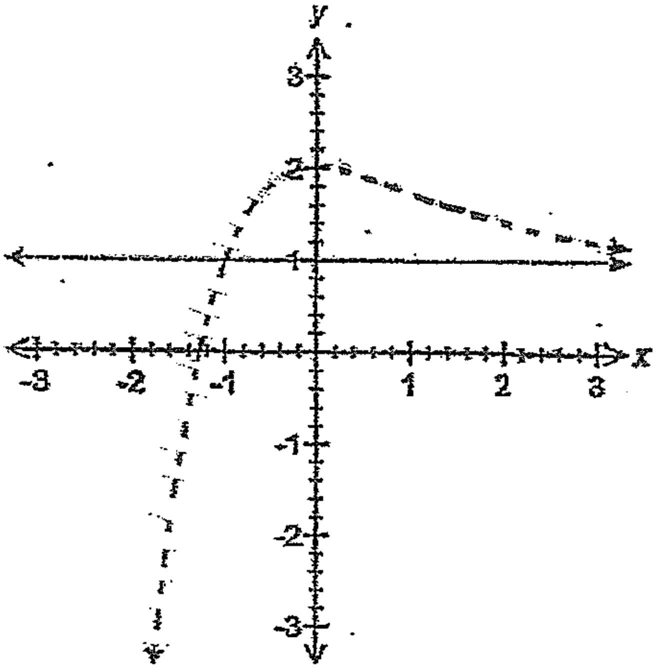


(γ)

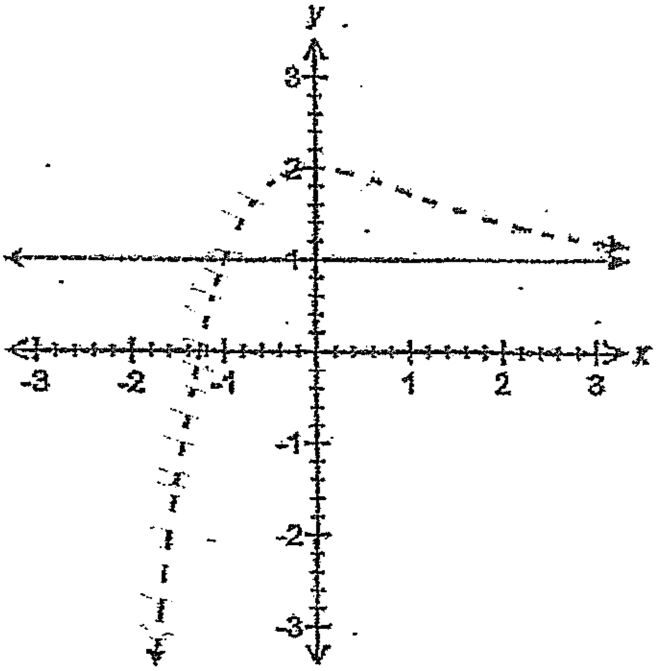


Template for Question 7 (d)

(i)

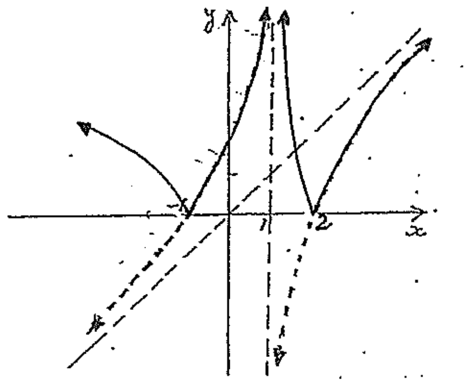


(ii)

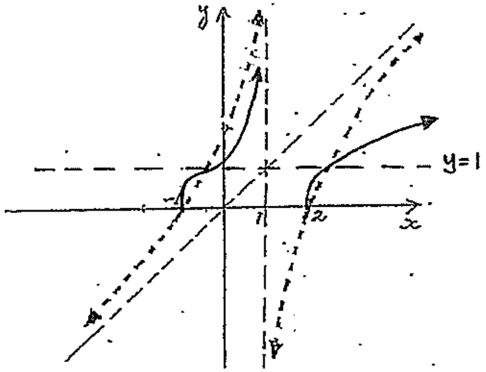


Template for Question 7 (c)(ii)

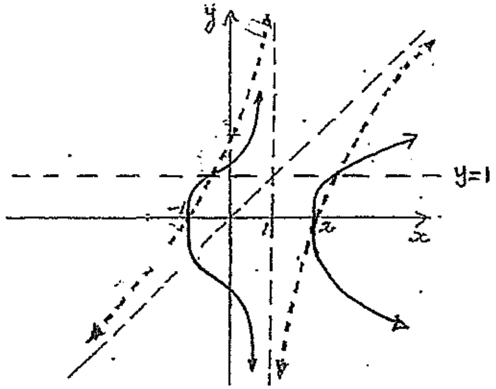
(α)



(β)

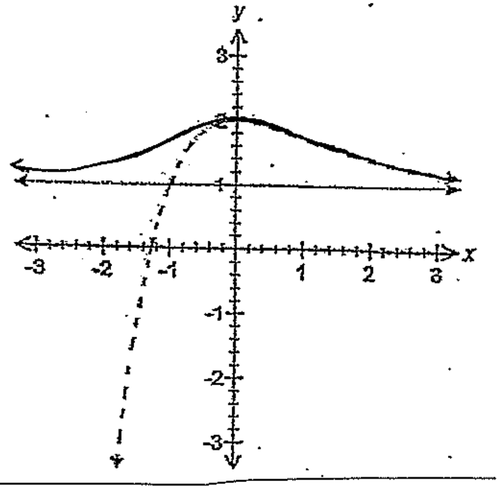


(γ)

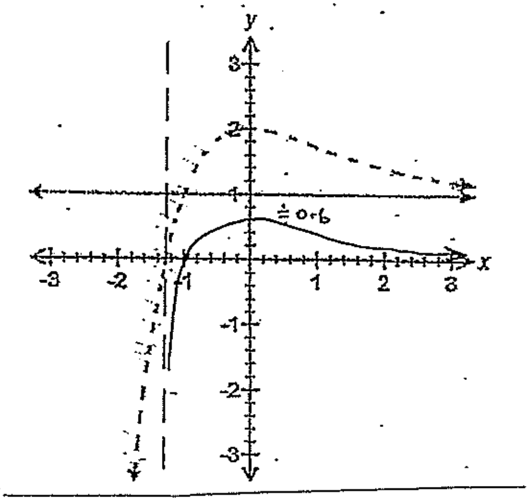


Template for Question 7 (d)

(I)



(II)



MATHEMATICS EXTENSION 2

Student Name____________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ☒ B ☒ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○

YEAR 12 MATHS EXT 2 TASK #2 SOLUTIONS

SECTION 1

1. C 2. A 3. D 4. A 5. B

$$-1+i = \sqrt{2}\left(\text{cis}\,\frac{3\pi}{4}\right)$$

$$\therefore z = \left[\sqrt{2}\left(\text{cis}\,\frac{3\pi}{4}\right)\right]^{1/9}$$

$$\therefore \arg z_1 = \frac{\pi}{12}$$

and other roots $z_2 \ldots z_9$ are evenly spaced about the Argand Diagram in increments of $\frac{\pi}{9}$

$$\therefore \arg z_1 \ldots z_9$$

$$= \frac{\pi}{12}, \frac{7\pi}{36}, \frac{11\pi}{36}, \frac{15\pi}{36}, \frac{19\pi}{36}, \frac{23\pi}{36}, \frac{27\pi}{36}, \frac{31\pi}{36}, \frac{35\pi}{36}$$

$$\therefore \arg z \neq \frac{29\pi}{36} \quad \therefore C$$

Let the roots be

$1-i,\ 1+i,\ \alpha$ → since real co-eff^s

Take the sum of these roots

$$\therefore 1-i+1+i+\alpha = -\frac{b}{a}$$

$$2+\alpha = 0$$

$$\therefore \alpha = -2 \quad \therefore A$$

3/ No vertical asymptotes if the denominator has no roots

i.e $\Delta < 0$

$$m^2+4n<0$$

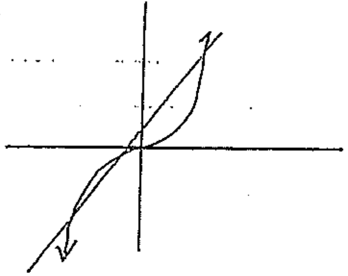
$$m^2 < -4n \quad \therefore D.$$

4/ A

5/ For $x^3+px-1=0$ to have 3 real roots

then $x^3 = 1-px$ has 3 real roots

Consider the graph:



and p is gradient of the line $y = 1-px$

since gradient must be positive

$p<0 \quad \therefore B$

SECTION II

Question 6.

(a) $w^2 - 2\bar{z}$

$$= (2i-1)^2 - 2(3+i)$$

$$= -4-4i+1-6-2i$$

$$= -9-6i$$

(b) (i) $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $z^7 = 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$$= 128\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$= 128\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$= -128\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= -64z$$

$$\therefore z^7 + 64z = 0$$

as req.

(c) $z^2 = 7-6\sqrt{2}i$

let $z = x+iy$

$$\therefore (x+iy)^2 = 7-6\sqrt{2}i$$

$$x^2-y^2+2xyi = 7-6\sqrt{2}i$$

Equating real / Imaginary part

$$x^2-y^2 = 7 \ldots (1)$$

$$2xy = -6\sqrt{2}$$

$$xy = -3\sqrt{2} \ldots (2)$$

From (2) $y = \frac{-3\sqrt{2}}{x}$

In (1) $x^2 - \frac{18}{x^2} = 7$

$$x^4-7x^2-18=0$$

$$(x^2-9)(x^2+2)=0$$

Since x is real

$$x^2 = 9$$

$$x = \pm 3$$

$$y = \mp\sqrt{2}$$

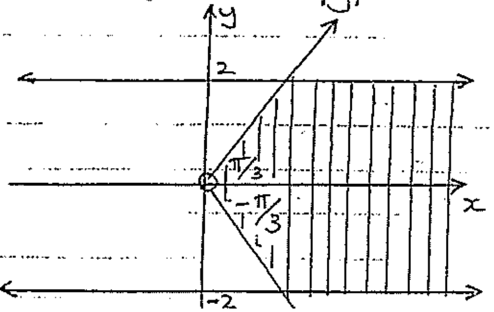
$$\therefore z = \pm(3-\sqrt{2}i)$$

(d) $z-\bar{z} = x+iy-(x-iy)$

$$= 2yi$$

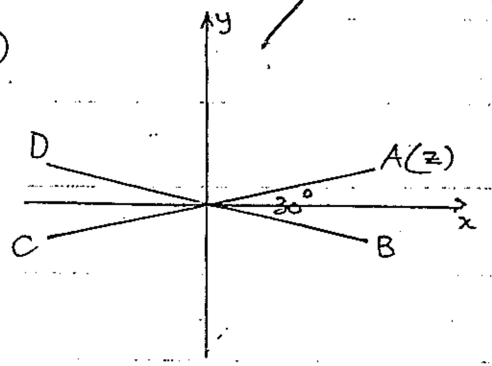
$$\therefore |z-\bar{z}| = |2y|$$

$$\therefore |2y| \le 4 \Rightarrow |y| \le 2$$



note: other solution
for ABCD anticlockwise

e)



$D = z \times \text{cis}\,150°$

$= z(\cos 150° + i\sin 150°)$

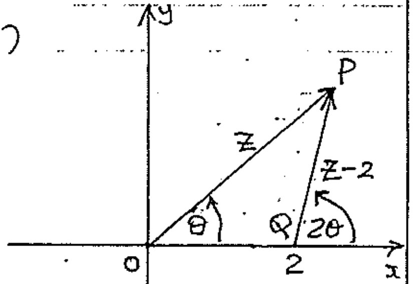
$= z\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ or $z\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$C = -z$

$B = -z\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= z\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$ or $z\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

f) (i)



$$\frac{\frac{1-e^x}{\cancel{e^x}}}{\frac{1+e^x}{\cancel{e^x}}}$$

(ii) If z is non-real, P does not lie on OQ

$O\hat{P}Q = \theta$ (exterior ∠ theorem of ΔOPQ)

∴ Since $O\hat{P}Q = P\hat{O}Q = \theta$

∴ ΔOPQ is isosceles.

∴ $OQ = PQ$

and $|z-2| = 2$

and therefore P lies on the circle with centre Q (2,0) and radius 2.

Question 7

(a)(i) $f(x) = \frac{e^x - 1}{e^x + 1}$

$f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} \times \frac{e^x}{e^x}$

$= \frac{1 - e^x}{1 + e^x}$

$-f(-x) = \frac{e^x - 1}{e^x + 1}$

∴ $f(x) = -f(-x)$

∴ an odd function

(ii) $f'(x) = \frac{(e^x+1)e^x - (e^x-1)e^x}{(e^x+1)^2}$

$= \frac{e^{2x} + e^x - e^{2x} + e^x}{(e^x+1)^2}$

$= \frac{2e^x}{(e^x+1)^2}$

Since $2e^x > 0$ and $(e^x+1)^2 > 0$

∴ $f'(x) > 0$ (2)

∴ $f(x)$ is always increasing

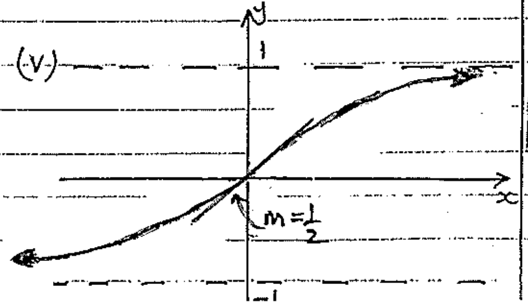
(iii) $f'(0) = \frac{2e^0}{(e^0+1)^2} = \frac{2}{4}$

$= \frac{1}{2}$

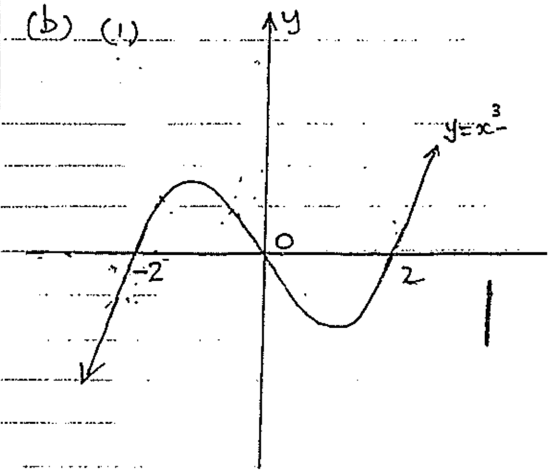
(iv) $x \to \infty$ $f(x) \to 1^-$

$x \to -\infty$ $f(x) \to -1^+$

(v)



(b) (i)



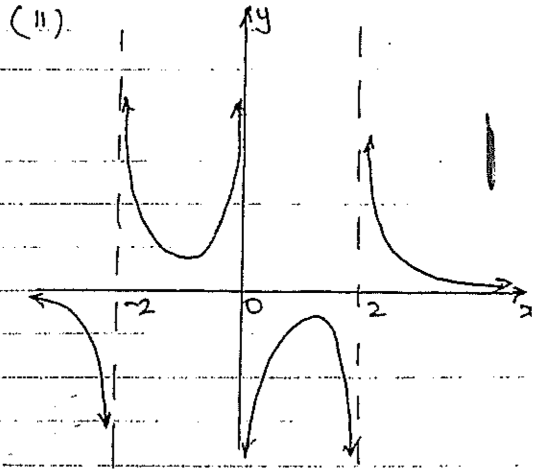
$y = x^3 - 4x$

$= x(x^2 - 4)$

$= x(x-2)(x+2)$

∴ intercepts $x = 0, \pm 2$

(ii)



(c) $f(x) = \dfrac{x^2-x-2}{x-1}$

$$\begin{array}{r|l} & x \\ \hline x-1 & x^2-x-2 \\ & \underline{x^2-x} \\ & \quad -2 \end{array}$$

$\therefore f(x) = x - \dfrac{2}{x-1}$

2

as $x \to \pm\infty$ $\quad \dfrac{2}{x-1} \to 0$

$\therefore y = x$ is an asymptote

(ii) see template

(d) see template

Question 8

(a) $\dfrac{x+7}{x^2(x+2)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$

$x+7 = Ax(x+2) + B(x+2) + Cx^2$

$(x=0)$ $7 = 0 + 2B + 0$

$\therefore B = \dfrac{7}{2}$

$=-2)$ $5 = 0 + 0 + 4C$

$\therefore C = \dfrac{5}{4}$

Equating co-eff's in x^2

$0 = A + C$

$0 = A + \frac{5}{4}$

$-\frac{5}{4} = A$

$\therefore$ RHS $= -\dfrac{5x}{4} + \dfrac{7}{2x^2} + \dfrac{5}{4(x+2)}$

(b) Product of roots $= \dfrac{e}{a}$

$= \dfrac{6-i}{3+2i}$

$= \dfrac{6-i}{3+2i} \times \dfrac{3-2i}{3-2i}$

$= \dfrac{18-12i-3i-2}{9+4}$

$= \dfrac{16-15i}{13}$

$= \dfrac{16}{13} - \dfrac{15i}{13}$

(c) If $P(-\frac{1}{2}) = 0$ then $2x+1$ is a factor

$$\begin{array}{r|l} & x^2-2x+3 \\ \hline 2x+1 & 2x^3-3x^2+4x+3 \\ & \underline{2x^3+x^2} \\ & \quad -4x^2+4x \\ & \quad \underline{-4x^2-2x} \\ & \qquad 6x+3 \\ & \qquad 6x+3 \end{array}$$

$\therefore P(x) = (2x+1)(x^2-2x+3)$

Now x^2-2x+3

$= x^2-2x+1+2$

$= (x-1)^2-(\sqrt{2}i)^2$

$= (x-1-\sqrt{2}i)(x-1+\sqrt{2}i)$

$\therefore P(x) = (2x+1)(x-1-\sqrt{2}i)(x-1+\sqrt{2}i)$

(d) let $P(y) = 2y^3-9y^2+12y+k$

$\therefore P'(y) = 6y^2-18y+12$

$= 6(y^2-3y+2)$

$= 6(y-2)(y-1)$

$\therefore$ Possible double roots are $y = 2$ or 1

If $y=2$ is a double root

$P(2) = 16-36+24+k = 0$

$4+k = 0$

$k = -4$

If $y=1$ is a double root

$P(1) = 2-9+12+k = 0$

$5+k = 0$

$k = -5$

$\therefore k = -4$ or -5

(e) let $y = \dfrac{1}{x^2}$

$\therefore x^2 = \dfrac{1}{y}$

$x = \dfrac{1}{\sqrt{y}}$

$\therefore$ equation becomes

$\left(\dfrac{1}{\sqrt{y}}\right)^3 + 2\left(\dfrac{1}{\sqrt{y}}\right)^2 - 3\left(\dfrac{1}{\sqrt{y}}\right) - 1 = 0$

$1 + 2\sqrt{y} - 3y - (\sqrt{3y})^3 = 0$

$2\sqrt{y} - y\sqrt{y} = 3y-1$

Squaring:

$4y - 4y^2 + y^3 = 9y^2 - 6y + 1$

$y^3 - 13y^2 + 10y - 1 = 0$

$\therefore x^3 - 13x^2 + 10x - 1 = 0$

is the new equation

f) (i) $w^k = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}$

If w^k is a solution then

$(w^k)^7 - 1 = 0$

$$\begin{aligned} LHS &= \cos 2\pi k + i\sin 2\pi k - 1 \\ &= 1 + 0 - 1 \\ &= 0 \\ &= RHS \end{aligned}$$ as req.

(ii) Since $z = 1$ is a root and w^k is a root then 7 roots are

$1, w, w^2, w^3, w^4, w^5, w^6$

and sum of roots $= -\frac{b}{a} = 0$

$\therefore 1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

$\therefore w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$

(iii) Complex roots are in conjugate pairs

i.e w and w^6
w^2 and w^5
w^3 and w^4

$\therefore w + w^6 = 2\cos\frac{2\pi}{7}$

$w^2 + w^5 = 2\cos\frac{4\pi}{7}$

$w^3 + w^4 = 2\cos\frac{6\pi}{7}$

$\therefore 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$
(from (ii))

$\therefore \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

as req.